

Computer Vision

MatLab intro & image Processing

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Spring of 2003

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Linear Algebra Review

Why do we need Linear Algebra?

- We will associate coordinates to
 - 3D points in the scene
 - 2D points in the CCD array
 - 2D points in the image
- Coordinates will be used to
 - Perform geometrical transformations
 - Associate 3D with 2D points

Sum:

A and B must have the same
dimensions

Example:

To enter the matrix

```
1 2
3 4
```

and store it in a variable a, do this:

```
>> a = [ 1 2; 3 4 ]
```

To redisplay the matrix, just type its name:

```
>> a
```

```
>> a + a
```

To store the result of this addition so that we can use it later:

```
>> s = a + a
```

Product:

$$C_{n \times p} = A_{n \times m} B_{m \times p}$$

$$c_{ij} = \sum_{k=1}^m a_{ik} b_{kj}$$

A and B must have compatible dimensions

$$A_{n \times n} B_{n \times n} \neq B_{n \times n} A_{n \times n}$$

Examples:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 17 & 29 \\ 19 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 17 & 10 \end{bmatrix}$$

Once you know how to enter and display matrices, it is easy to compute with them. First we will square the matrix a :

```
>> a * a
```

Now

```
>> b = [ 1 2; 0 1 ]
```

Then we compute the product ab :

```
>> a*b
```

we compute the product in the other order:

```
>> b*a
```

Notice that the two products are different: matrix multiplication is noncommutative.

□ Transpose:

$$C_{m \times n} = A^T_{n \times m}$$

$$(A + B)^T = A^T + B^T$$

$$c_{ij} = a_{ji}$$

$$(AB)^T = B^T A^T$$

$$\square\square \quad A^T = A \quad A \text{ is symmetric} \square\square$$

Examples:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \\ 3 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$

```
>> s = a + b
```

```
s =
     2     4
     3     5
```

To get the transpose of s:

```
>> s'
ans =
     2     3
     4     5
```

◻ determinant: A must be square

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example: $\det \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = 2 - 15 = -13$

```
>> det(s)
```

```
ans -2
```

Inverse:

A must be square

$$A_{n \times n} A^{-1}_{n \times n} = A^{-1}_{n \times n} A_{n \times n} = I$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example: $\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 28 & 0 \\ 0 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
>> inv(s)
```

To check that this is correct, we compute the product of s and its inverse:

```
>> s * inv(s)
```

The result is the unit, or identity matrix. We can also write the computation as

```
>> s/s
```

```
>> s\s
```

which is the same as

```
>> inv(s) * s
```

Not all matrices can be inverted, or used as the denominator in matrix division:

```
>> c = [ 1 1; 1 1 ]
```

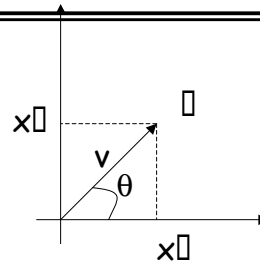
```
>> inv(c);
```

A matrix can be inverted if and only if its determinant is nonzero:

```
>> det(a)
```

```
>> det(c)
```

$$\mathbf{v} = (x_1, x_2)$$



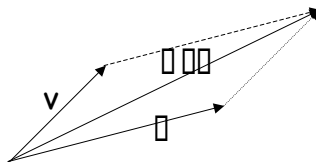
□ magnitude: $\|\mathbf{v}\| = \sqrt{x_1^2 + x_2^2}$

□□ $\|\mathbf{v}\| = 1$ □ \mathbf{v} □s a □□ □□ ve□to□

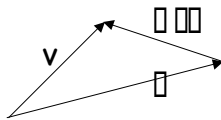
$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{x_1}{\|\mathbf{v}\|}, \frac{x_2}{\|\mathbf{v}\|} \right) \text{ □s a unit ve□to□}$$

□ orientation: $\theta = \tan^{-1} \left(\frac{x_2}{x_1} \right)$

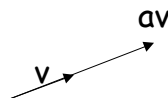
$$\mathbf{v} + \mathbf{w} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

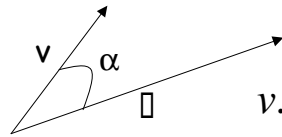


$$\mathbf{v} - \mathbf{w} = (x_1, x_2) - (y_1, y_2) = (x_1 - y_1, x_2 - y_2)$$



$$a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2)$$



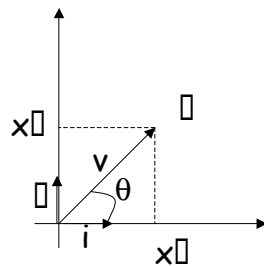


$$v \cdot w = (x_1, x_2) \cdot (y_1, y_2) = x_1 y_1 + x_2 \cdot y_2$$

The inner product is a scalar

$$v \cdot w = (x_1, x_2) \cdot (y_1, y_2) = \|v\| \cdot \|w\| \cos \alpha$$

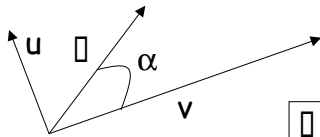
$$v \cdot w = 0 \Leftrightarrow v \perp w$$



$$\begin{aligned} \mathbf{i} &= (1,0) & \|\mathbf{i}\| &= 1 & \mathbf{i} \cdot \mathbf{j} &= 0 \\ \mathbf{j} &= (0,1) & \|\mathbf{j}\| &= 1 & & \end{aligned}$$

$$\mathbf{v} = (x_1, x_2)$$

$$\mathbf{v} = x_1 \cdot \mathbf{i} + x_2 \cdot \mathbf{j}$$



$$u = v \times w$$

The cross product is a vector

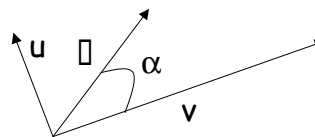
• magnitude: $\|u\| = \|v \times w\| = \|v\| \|w\| \sin \alpha$

• orientation: $u \perp v \implies u \cdot v = (v \times w) \cdot v = 0$
 $u \perp w \implies u \cdot w = (v \times w) \cdot w = 0$

$$\begin{aligned} \mathbf{i} &= (1, 0, 0) & \|\mathbf{i}\| &= 1 \\ \mathbf{j} &= (0, 1, 0) & \|\mathbf{j}\| &= 1 & \mathbf{i} \cdot \mathbf{j} &= 0, \mathbf{i} \cdot \mathbf{k} = 0, \mathbf{j} \cdot \mathbf{k} = 0 \\ \mathbf{k} &= (0, 0, 1) & \|\mathbf{k}\| &= 1 \end{aligned}$$

$$\mathbf{u} = \mathbf{v} \times \mathbf{w} = (x_1, x_2, x_3) \times (y_1, y_2, y_3)$$

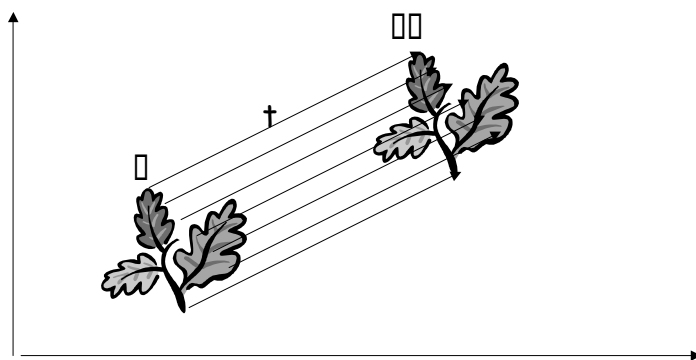
$$\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$



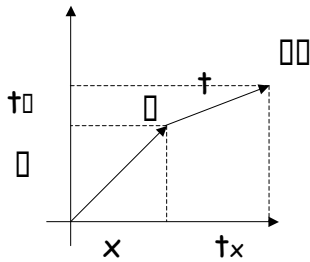
$$= (x_2 y_3 - x_3 y_2) \mathbf{i} + (x_3 y_1 - x_1 y_3) \mathbf{j} + (x_1 y_2 - x_2 y_1) \mathbf{k}$$

2D Geometrical Transformations

2D Translation



2D Translation Equation

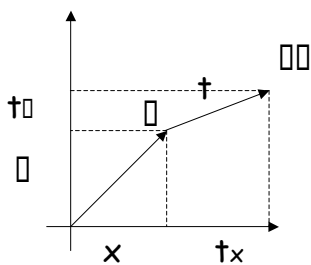


$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' = (x + t_x, y + t_y) = \mathbf{P} + \mathbf{t}$$

2D Translation using Matrices



$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- Multiply the coordinates by a non-zero scalar and add an extra coordinate equal to that scalar. For example,

$$(x, y) \rightarrow (x \cdot z, y \cdot z, z) \quad z \neq 0$$

$$(x, y, z) \rightarrow (x \cdot w, y \cdot w, z \cdot w, w) \quad w \neq 0$$

- □ □ □ E: □□ the scala□ is □□ the□e is no need □□ the multipli□ation□

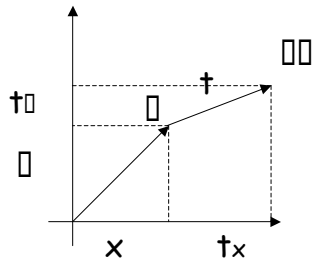
Back to Cartesian Coordinates:

- □ ivide □□ the last □□o□dinate and eliminate it□□o□ example□

$$(x, y, z) \quad z \neq 0 \rightarrow (x/z, y/z)$$

$$(x, y, z, w) \quad w \neq 0 \rightarrow (x/w, y/w, z/w)$$

2D Translation using Homogeneous Coordinates



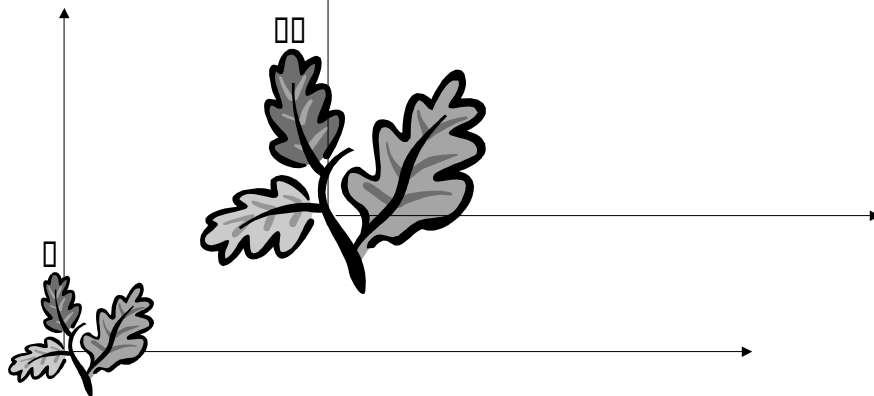
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

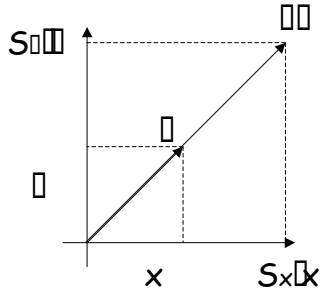
$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{T} \cdot \mathbf{P}$$

Scaling



Scaling Equation



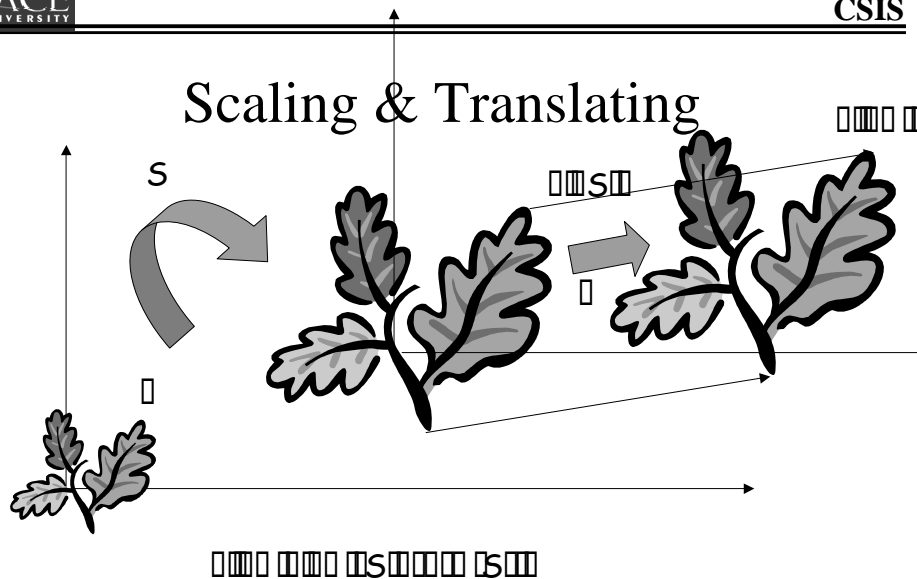
$$P = (x, y) \rightarrow (x, y, 1)$$

$$P' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$P' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = S \cdot P$$

Scaling & Translating



Scaling & Translating

Translation followed by scaling

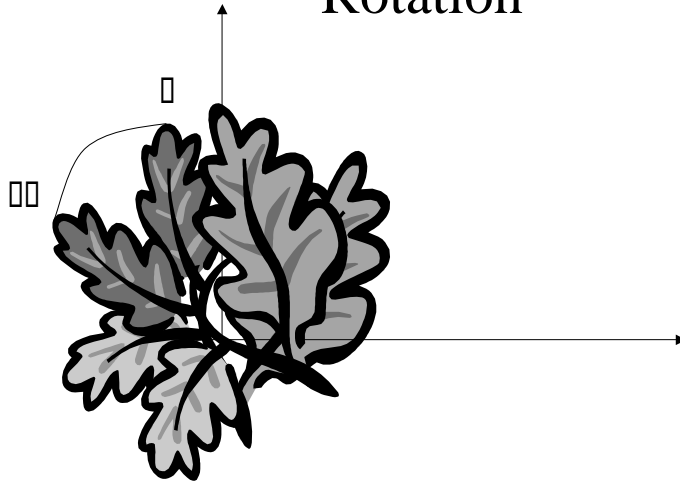
$$\begin{aligned}
 \mathbf{P}'' &= \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \\
 &= \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}
 \end{aligned}$$

Translating & Scaling ≠ Scaling & Translating

Scaling followed by translation

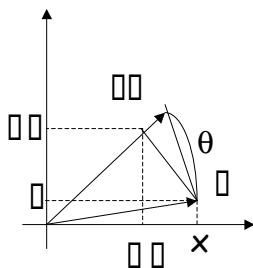
$$\begin{aligned}
 \mathbf{P}'' &= \mathbf{S} \cdot \mathbf{T} \cdot \mathbf{P} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \\
 &= \begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + s_x t_x \\ s_y y + s_y t_y \\ 1 \end{bmatrix}
 \end{aligned}$$

Rotation



Rotation Equations

Counterclockwise rotation by an angle θ



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{R} \cdot \mathbf{P}$$

Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is 2×2 \rightarrow 4 elements

But here there is only 1 degree of freedom: θ

The 4 elements must satisfy the following 2 constraints:

$$\mathbf{R} \cdot \mathbf{R}^T = \mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

Scaling, Translating & Rotating



degrees of freedom

1×1 2×2
 2×2 2×2 2×2
 2×2 2×2 2×2 2×2 2×2



$2 \times 2 \neq 2 \times 2 \neq 2 \times 2$

3D Rotation

Rotation around the coordinate axes:

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3D Rotation (axis & angle)

$\mathbf{n} = [n_1 \ n_2 \ n_3]^T$, angle θ

$$\mathbf{R} = \mathbf{I} \cos \theta + \mathbf{I} (1 - \cos \theta) \begin{bmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^2 & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & n_3^2 \end{bmatrix} + \sin \theta \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$